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A Study of Operands in Terms of Maximal Generalized Orbits

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An operand X of a monoid S is called saturated if every generalized orbit in X is contained in a union of others. Every operand has a natural decomposition as a union of an operand admitting an irredundant cover by maximal generalized orbits and of a saturated operand. There is a descending chain of suboperands of an operand which leads to the definition of the saturation length of an operand. S has no saturated operands if and only if S satisfies the ascending chain condition on orbits.

Let S be a monoid and let X be a left S-operand. The identity of S will always act as the identity on X. An operand X will be called a *locally cyclic operand* or a *generalized orbit* if for every $x, y \in X$ there exists $z \in X$ such that $x, y \in Sz$. Call a generalized orbit *proper* if it is not itself an orbit. It is well known [1] that an arbitrary operand is the union of its orbits; hence it is also the union of its generalized orbits. Although in general there need not be maximal orbits, we have the following proposition.

PROPOSITION 0.1. *If X is a left S-operand and $x \in X$, then x is contained in some maximal generalized orbit $M \subset X$.*

Proof. The set of generalized orbits containing x is nonempty, since Sx is in it, and the union of a chain (under inclusion) of generalized orbits is easily seen to be a generalized orbit also. Hence by Zorn's lemma, this collection has a maximal element, the desired M.

1. SATURATED OPERANDS AND COVERS

If a left S-operand X can be written $X = \bigcup \{M_i : i \in I\}$ where each M_i is a maximal generalized orbit of X and where no M_i is contained in the

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union of the others, then we say that X admits an *irredundant cover* by maximal generalized orbits. On the other hand, if X is such that every maximal generalized orbit of X is contained in a union of other maximal generalized orbits, then call X *saturated.*

LEMMA. *Suppose X is a left S-operand such that $X = A \cup B$ for sub-operands A and B.*

(i) *Every generalized orbit of X lies in either A or B.*

(ii) *If $A = S(X \backslash B)$, then A is a union of maximal generalized orbits of X.*

Proof. (i) If $M \subset X$ is a generalized orbit of X and $M \not\subset A$, then there is a $b \in M \cap B$ with $b \notin A$. For every $m \in M$ there is $x_m \in M$ such that $m, b \in Sx_m$. Since $b \notin A$, $x_m \notin A$. Thus $x_m \in B$ for all $m \in M$, and $M \subset B$.

(ii) If $a \in A$, then there is an $x \in X \backslash B$ such that $a \in Sx$.

Let M be a maximal generalized orbit of X such that $x \in M$. Thus $a \in M$, and since $M \not\subset B$, $M \subset A$ by (i). Hence A is a union of maximal generalized orbits of X.

PROPOSITION 1.1. *If a left S-operand X can be covered by a finite number of maximal generalized orbits, then this cover is irredundant.*

Proof. It suffices to show that if $M, M_1, \ldots, M_n$ are maximal generalized orbits of X such that $M \subset M_1 \cup \cdots \cup M_n$, then $M = M_i$ for some $i = 1, \ldots, n$. Let $M_1' = M_2 \cup \cdots \cup M_n$. Then M_1' is a suboperand of X and $M \subset M_1 \cup M_1'$. By the lemma either $M \subset M_1$ or $M \subset M_1'$. Either $M = M_1$ by maximality, or we repeat the argument. In a finite number of steps we find that $M = M_i$ for some i.

PROPOSITION 1.2. (i) *If X is a left S-operand, if X admits an irredundant cover by maximal generalized orbits, and if M is a member of such a cover, then M cannot be covered by a union of other maximal generalized orbits.*

(ii) *If X is a saturated left S-operand, then X admits no irredundant cover by maximal generalized orbits.*

Proof. (i) clearly implies (ii).

To prove (i) let $X = M \cup (\bigcup \{N_j : j \in J\})$ be an irredundant covering of X by maximal generalized orbits. Suppose that $M \subset \bigcup \{M_i : i \in I\}$ where each M_i is a maximal generalized orbit distinct from M. Since $\bigcup \{N_j : j \in J\}$ is a suboperand of X and since $M_i \subset X = M \cup (\bigcup \{N_j : j \in J\})$ for all $i \in I$, we conclude by the lemma that $M_i \subset \bigcup \{N_j : j \in J\}$ for all $i \in I$. Hence $M \subset \bigcup \{M_i : i \in I\} \subset \bigcup \{N_j : j \in J\}$ contrary to the hypothesis that the given cover of X is irredundant. Hence M cannot be covered by a union of other maximal generalized orbits.

COROLLARY 1.2.1. *If a left S-operand admits an irredundant cover by maximal generalized orbits, then this cover is unique.*

Proof. Let X be a left S-operand, and let $U = \{x \in X : x$ is contained in a unique maximal generalized orbit of $X\}$. Let M be a maximal generalized orbit of X. By Proposition 1.2, if M is a member of an irredundant cover of X, then $M \cap U \neq \varnothing$. Likewise if $M \cap U \neq \varnothing$, then M must be included in any cover of X. Thus M is a member of an irredundant cover of X if and only if $M \cap U \neq \varnothing$.

PROPOSITION 1.3. *A left S-operand X is saturated if and only if every $x \in X$ lies in two distinct maximal generalized orbits.*

Proof. If X is saturated, and if M is a maximal generalized orbit with $x \in M$, then $M \subset \bigcup \{M_i : i \in I\}$ where each M_i is a maximal generalized orbit distinct from M. Thus $x \in M_i$ for some i and $M \neq M_i$.

Conversely, let M be a maximal generalized orbit. Then for every $x \in M$ there is a maximal generalized orbit $M_x \neq M$ with $x \in M_x$. Hence $M \subset \bigcup \{M_x : x \in M\}$ is a covering of M by a union of other maximal generalized orbits. Thus X is saturated.

COROLLARY 1.3.1. *A left S-operand X is saturated if and only if for each $x \in X$ there exist $y, z \in X$ such that $x \in Sy \cap Sz$ and such that no orbit of X contains both y and z.*

Proof. If the condition holds, then y and z are contained in distinct maximal generalized orbits, and x is contained in both. Thus X is saturated by Proposition 1.3.

Conversely, if for every $y, z \in X$ such that $x \in Sy \cap Sz$ there exists $w \in X$ such that $y, z \in Sw$, then $M = \{Sy : y \in X$ and $x \in Sy\}$ is a generalized orbit and is the unique maximal generalized orbit of X containing x. Hence X is not saturated.

THEOREM 1.4. *For every left S-operand X exactly one of the following is true.*

(i) *X admits an irredundant cover by maximal generalized orbits.*

(ii) *X is saturated.*

(iii) *Neither* (i) *nor* (ii) *hold and $X = X' \cup X''$ where X' and X'' are suboperands such that X' admits an irredundant cover by maximal generalized* *orbits, X'' is saturated, and X' and X'' are each generated by the complement of the other. Furthermore, X' and X'' are uniquely determined by these properties.*

Proof. Let $U = \{x \in X : x$ is contained in exactly one maximal generalized orbit of $X\}$. By Proposition 1.3 X is saturated if and only if $U = \varnothing$.

If $U \neq \varnothing$, we observe that if $x \in U$ and $x \in Sy$ for some $y \in X$, then $y \in U$. Thus if N is a maximal generalized orbit of X such that $N \cap U \neq \varnothing$, we may take $x \in N \cap U$ and find for every $n \in N$ a $y_n \in N \cap U$ such that $x, n \in Sy_n$; hence $N = S(N \cap U)$. Let $X' = SU$. Let $\{N_j : j \in J\}$ be the set of maximal generalized orbits of X such that $N_j \cap U \neq \varnothing$. Then for all $j \in J$, $N_j = S(N_j \cap U) \subset X'$ and since $U \subset \bigcup \{N_j : j \in J\}$, in fact $X' \subset \bigcup \{N_j : j \in J\}$. This cover is irredundant since, if $j_0 \in J$ and if $x \in N_{j_0} \cap U$, then $x \notin \bigcup \{N_j : j_0 \neq j \in J\}$.

If $X = X'$, then (i) is true. By Corollary 1.2.1 and the above, if X admits an irredundant cover by maximal generalized orbits, then $X = X'$. Hence $X = X'$ if and only if (i) holds.

If $X' \neq X$, set $V = X \backslash X'$, and let $X'' = SV$. Let N be a maximal generalized orbit of X such that $x \in N \cap V \neq \varnothing$. For every $n \in N$ there exists $y_n \in N$ such that $x, n \in Sy_n$. Since $x \notin X'$, it must be that $y_n \notin X'$. Thus $y_n \in V$ so that $N = S(N \cap V) \subset X''$. Clearly N is also a maximal generalized orbit of X'', and in fact $X'' = \bigcup \{N : N$ is a maximal generalized orbit of X and $N \cap V \neq \varnothing\}$. Since $U \cap V = \varnothing$, every element of V (and hence every element of X'') is contained in two distinct maximal generalized orbits of X and thus in two distinct maximal generalized orbits of X''. Therefore X'' is saturated.

Suppose now that $X = Y' \cup Y''$ where Y' and Y'' are suboperands of X such that Y' admits an irredundant covering by maximal generalized orbits, Y'' is saturated, and Y' and Y'' are each generated by the complement of the other. Then by the lemma Y' and Y'' are unions of maximal generalized orbits of X and every generalized orbit of X lies in either Y' of Y''.

If $x \in Y''$, then there exist maximal generalized orbits M and M' of Y'' such that $x \in M \cap M'$ and $M \neq M'$. Without loss of generality we may assume that M is a maximal generalized orbit of X. Let M'' be a maximal generalized orbit of X such that $M' \subset M''$. Then $M \neq M''$, since otherwise $M = M'$. Thus $x \notin U$, so that $U \cap Y'' = \varnothing$. But then $U \subset Y'$, which implies that $X' \subset Y'$.

Let $W = \{y \in Y' : y$ is contained in a unique maximal generalized orbit of $Y'\}$. Since Y' admits an irredundant cover by maximal generalized orbits of Y', then $Y' = SW$. Let $w \in W$. Then there is a $w' \in X \backslash Y''$ such that $w \in Sw'$. Thus $w' \in W$. Now there is a maximal generalized orbit N of X such that $w' \in N \subset Y'$. If M is a maximal generalized orbit of X such that $w' \in M$, then either $M \subset Y'$ or $M \subset Y''$. But since $w' \notin Y''$, it must be that

$M \subset Y'$. But then since $w' \in W$, it must be that $M = N$. Thus $w' \in U$, so that $W \subset SU = X'$. This gives $Y' \subset X'$, which together with the above, gives $Y' = X'$.

Then $Y'' = S(X \backslash Y') = S(X \backslash X') = X''$.

2. Saturation Length

For a left S-operand X define a descending chain of suboperands indexed by the ordinals as follows. Set $X_0 = X$. Having defined X_α we set $\mathscr{M}_\alpha = \{M \subset X_\alpha : M$ is a maximal generalized orbit in $X_\alpha\}$ and let $X_{\alpha+1} = \{x \in X : x \in M \cap M'$ where $M, M' \in \mathscr{M}_\alpha$ and $M \neq M'\}$. If α is a limit ordinal, let $X_\alpha = \bigcap \{X_\beta : \beta < \alpha\}$.

Proposition 2.1. *If* $X = X_0 \supset X_1 \supset \cdots \supset X_\alpha \supset X_{\alpha+1} \supset \cdots$ *is the descending chain defined above, then for some ordinal* α *this chain terminates in the sense that either* $X_\alpha = \varnothing$ *or* $X_\alpha = X_{\alpha+1} \neq \varnothing$. *In the latter case* X_α *is saturated.*

Proof. The existence of such an α follows from cardinality considerations, and in fact it must be that $\operatorname{card}(\alpha) \leqslant \operatorname{card}(X)$. If $X_\alpha = X_{\alpha+1} \neq \varnothing$, then by Proposition 1.3 X_α is saturated.

Definition. Call the least ordinal α such that $X_\alpha = \varnothing$ or $X_\alpha = X_{\alpha+1} \neq \varnothing$ the *saturation length* of X.

This descending chain of operands gives information about the way the maximal generalized orbits of X are joined together.

Theorem 2.2. *If* X *is a left* S*-operand, if* α *is the saturation length of* X*, and if* $X = X' \cup X''$ *as in Theorem* 1.4, *then*

(i) $X_\alpha \supset X''$; *hence if* $X_\alpha = \varnothing$, *then* X *admits an irredundant cover by maximal generalized orbits.*

(ii) $X' = S(X \backslash X_\alpha)$.

(iii) *It can happen that* $X'' = \varnothing$, *but* $X_\alpha \neq \varnothing$.

Proof. Recall from the proof of Theorem 1.4 that $U = \{x \in X : x$ lies in exactly one maximal generalized orbit of $X\}$, $X' = SU$, $V = X \backslash X'$, and $X'' = SV$.

Recall also that if $M \subset X$ is a maximal generalized orbit such that $M \cap V \neq \varnothing$, then $M = \bigcup \{Sv : v \in M \cap V\} \subset X''$ and $X'' = \bigcup \{M : M$ is a maximal generalized orbit of X with $M \cap V \neq \varnothing\}$.

(i) If $X'' = V = \varnothing$, there is nothing to prove. If $X'' \supset V \neq \varnothing$, then we show that if M is a maximal generalized orbit of X such that $M \cap V \neq \varnothing$, then $M \subset X_\sigma$ for all ordinals σ. This is trivially true for $X_0 = X$. Suppose that it is true for all ordinals $\tau < \sigma$. If σ is a limit ordinal, then $M \subset \bigcap\{X_\tau : \tau < \sigma\} = X_\sigma$. If $\sigma = \tau + 1$, let M be a maximal generalized orbit of X such that $M \cap V \neq \varnothing$. For every $v \in M \cap V$ there exists a maximal generalized orbit M_v of X such that $v \in M \cap M_v$ and $M \neq M_v$. Since $\tau < \sigma$, $M, M_v \subset X_\tau$ for all $v \in M \cap V$, and being maximal generalized orbits of X, these are maximal generalized orbits of X_τ. Hence $M \cap V \subset X_{\tau+1} = X_\sigma$, which implies that $M \subset X_{\tau+1}$. Therefore by induction $M \subset X_\sigma$ for all ordinals σ for every maximal generalized orbit of X such that $M \cap V \neq \varnothing$. Thus $X'' \subset X_\sigma$ for all σ, so that, in particular, $X'' \subset X_\alpha$.

(ii) Observe that $U = X_0 \backslash X_1 \subset X \backslash X_\alpha$ so that $X' = SU \subset S(X \backslash X_\alpha)$. However, since $X'' \subset X_\alpha$, one finds that $X \backslash X_\alpha \subset X \backslash X'' \subset X'$, which gives the reverse inclusion.

(iii) It suffices to show that X may admit an irredundant cover by maximal generalized orbits (so that $X'' = \varnothing$) and yet that $X_\alpha \neq \varnothing$. This will be seen in the following example.

EXAMPLE 2.3. Let the monoid be $N = \{0, 1, 2, \ldots\}$ under addition. Let $X = \{(x_0, x_1, x_2, \ldots) : x_i \in Z$ for all i, $x_i \geqslant 0$ for $i \geqslant 1$, $x_i = 0$ for almost all i, and if $x_n \neq 0$, then $x_k \neq 0$ for $0 < k \leqslant n$ and x_k must be odd for $0 \leqslant k \leqslant n - 2\}$. In words, X is the set of all finite sequences of integers, all except the first of which is nonnegative, such that a zero entry occurs before a nonzero entry only if it occurs in the 0th place and such that nonzero even integers can occur only in the last two nonzero places. We define the action of N on X by $1 \cdot (x_0, \ldots, x_{m-1}, x_m, 0, \ldots) = (x_0, \ldots, x_{m-1}, x_m - 1, 0, \ldots)$ where $x_m \neq 0$. In general

$$n \cdot (x_0, x_1, \ldots, x_m, 0, 0, \ldots)$$

$$= \begin{cases} (x_0, \ldots, x_k + \cdots + x_m - n, 0, \ldots) \text{ if there is a} \\ \quad k \geqslant 1 \text{ such that } x_{k+1} + \cdots + x_m < n \leqslant x_k + \cdots + x_m, \\ (x_0 + \cdots + x_m - n, 0, \ldots) \text{ if } x_1 + \cdots + x_m < n. \end{cases}$$

One easily checks that this makes X a left N-operand.

Let Σ be the set of all sequences $\sigma = (x_0, x_1, x_2, \ldots)$ of integers such that $x_i \geqslant 0$ for all $i \geqslant 1$, such that $x_n \neq 0$ implies $x_k \neq 0$ for all $0 < k \leqslant n$, and such that either this is an infinite sequence of odd integers or there is a last nonzero integer x_n, $x_n = 1$, and x_k is odd if $0 \leqslant k \leqslant n - 2$. For $\sigma \in \Sigma$ as above, define $M(\sigma)$ as follows:

(i) if $\sigma = (x_0, \ldots, x_n, \ldots)$ is an infinite sequence of odd integers, then $M(\sigma) = \bigcup\{N \cdot (x_0, x_1, \ldots, x_n, 0, \ldots) : n = 0, 1, 2, \ldots\}$.

(ii) if σ has a last nonzero entry $x_n = 1$, then

$$M(\sigma) = N \cdot (x_0, \ldots, x_{n-1}, 0, \ldots) \cup \{(x_0, \ldots, x_{n-1}, k, 0, \ldots) : k = 1, 2, 3, \ldots\}.$$

Then one can verify that $\{M(\sigma) : \sigma \in \Sigma\}$ is the set of distinct maximal generalized orbits of X. This follows from the observation that $(x_0, \ldots, x_k, 0, \ldots)$ and $(y_0, \ldots, y_m, 0, \ldots)$ with $x_k \neq 0$, $y_m \neq 0$, and $k \leqslant m$ are in the same orbit only if $x_i = y_i$ for $i = 0, \ldots, k-1$, and then one is in the orbit generated by the other.

Let $\gamma = (x_0, \ldots, x_r, 1, 0, \ldots)$ where $r \geqslant 0$ and x_r is even be considered either as an element of X or of Σ. Then $\gamma \in X$ lies in exactly one maximal generalized orbit, namely $M(\gamma)$, since if $\gamma \in N \cdot (y_0, \ldots, y_p, 0, \ldots)$, then $p = r + 1$, and $y_i = x_i$ for $i = 0, \ldots, r$.

Now let $\beta = (x_0, \ldots, x_n, 0, \ldots) \in X$ with $x_n \neq 0$. If $n = 0$ or if $n > 0$ and x_{n-1} is odd, let

$$\sigma(\beta) = \begin{cases} (x_0, \ldots, x_n, 1, 0, \ldots) & \text{if } x_n \text{ is even,} \\ (x_n, \ldots, x_n + 1, 1, 0, \ldots) & \text{if } x_n \text{ is odd.} \end{cases}$$

If $n > 0$ and x_{n-1} is even, set $\sigma(\beta) = (x_0, \ldots, x_{n-1}, 1, 0, \ldots)$. Then $\beta \in M(\sigma(\beta))$ for all $\beta \in X$. Since each $\sigma(\beta)$, thought of as an element of X, lies in a unique maximal generalized orbit, namely $M(\sigma(\beta))$, we find that $X = \bigcup\{M_\sigma : \sigma \in \Sigma$ and $\sigma = \sigma(\beta)$ for some $\beta \in X\}$ is an irredundant cover of X by maximal generalized orbits. This says that in this case $X = X'$ so that $X'' = \varnothing$.

We find now that X_1 is saturated (i.e., $\alpha = 1$ here). First we see that $(x_0, \ldots, x_{n-1}, x_n, 0, \ldots) \in X$ with $x_n \neq 0$ lies in a unique maximal generalized

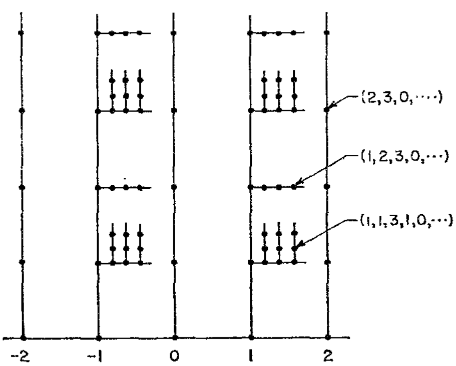

FIGURE 1

orbit if and only if x_{n-1} is even. Hence $X_1 = \{(x_0, \ldots, x_{n-1}, x_n, 0, \ldots) \in X : x_n \neq 0$ and x_{n-1} is odd$\}$. If $\beta = (x_0, \ldots, x_{n-1}, x_n, 0, \ldots) \in X_1$ with $x_n \neq 0$, let $M = M((x_0, \ldots, x_{n-1}, 1, 0, \ldots))$ and $M' = M((x_0, \ldots, x_{n-1}, 2x_n + 1, 1, 0, \ldots))$. Then $M, M' \subset X_1$ so that these are maximal generalized orbits of X_1, $\beta \in M \cap M'$, and $M \neq M'$ since $(x_0, \ldots, x_{n-1}, 2x_n + 1, 1, 0, \ldots) \notin M$ and $(x_0, \ldots, x_{n-1}, 2x_n + 2, 0, \ldots) \notin M'$. Hence by Proposition 1.3, X_1 is saturated.

The nature of X is seen from the part of it drawn in Fig. 1.

3. Ascending Chain Condition

One might hope that saturated operands were the exception rather than the rule; however we have the following. Say that S *satisfies the ascending chain condition on left orbits* if every left S-operand satisfies the ascending chain condition on orbits.

Theorem 3.1. *The following are equivalent for a monoid S:*

(i) *S has no saturated left operands.*

(ii) *S satisfies the ascending chain condition for left orbits.*

(iii) *Every left generalized orbit over S is an orbit.*

(iv) *Every left S-operand is an irredundant union of maximal orbits.*

Proof. (ii) trivially implies (iii). (iii) implies (iv) via Proposition 0.1, the irredundancy following since if an orbit is contained in the union of other orbits, it must be contained in one of them. (iv) implies (i) via Proposition 1.2 (ii).

It remains to show that (i) implies (ii). Suppose that S has a left operand X which is the union of a proper ascending chain of orbits, say, $Sx_0 \subsetneqq Sx_1 \subsetneqq Sx_2 \subsetneqq \cdots \subset X$. Let $T = \{(t_1, t_2, \ldots) : t_i = 0, 1$ for $i = 1, 2, \ldots\}$. The set $X \times T$ becomes a left S-operand via the action $s \cdot (x, t) = (sx, t)$ for $s \in S$, $x \in X$, and $t \in T$. Define a relation $\sim$ on $X \times T$ by $(x, t) \sim (x', t')$ if and only if $x = x'$ and $t_i = t_i'$ for $i = 1, \ldots, n$ where $x \in X_n$ but $x \notin X_{n-1}$. (There is no condition on t and t' if $x = x' \in X_0$.) The relation $\sim$ is a congruence on the operand $X \times T$. Let $Y = (X \times T)/\sim$ and denote the class of (x, t) in Y by $[x, t]$. To see that Y is a saturated left S-operand it suffices by Corollary 1.3.1 to show that for every $[x, t] \in Y$ there exist $[x', t']$ and $[x'', t'']$ in Y such that $[x, t] \in S[x', t'] \cap S[x'', t'']$ and such that $[x', t']$ and $[x'', t'']$ do not lie in a common orbit of Y. If $x \in Sx_n$ and $x \notin Sx_{n-1}$ and $t = (t_1, t_2, \ldots, t_n, \ldots)$, then take $x' = x'' = x_{n+1}$, $t' = (t_1, \ldots, t_n, 0, \ldots)$, and $t'' = (t_1, \ldots, t_n, 1, 0, \ldots)$. Then

$$[x', t'] \neq [x'', t''] \qquad \text{and} \qquad [x, t] \in S[x', t'] \cap S[x'', t''].$$

Observe now that $[x', t'] \in S[x''', t''']$ if and only if $x' \in Sx'''$ and $t_i' = t_i'''$ for $i = 1, \ldots, n+1$, so that $t'''_{n+1} = 0$. Likewise $[x'', t''] \in S[x''', t''']$ implies that $t'''_{n+1} = 1$. Hence $[x', t']$ and $[x'', t'']$ do not lie in a common orbit of Y, so that Y is saturated. This contradicts (i), and therefore proves that (i) implies (ii).

PROPOSITION 3.2. *If S is a monoid satisfying the ascending chain condition on orbits, then no orbit X of S contains a proper suborbit which is isomorphic to X.*

Proof. If $X = Sx$ for $x \in X$ and there is a $u \in X$ such that $Su \subsetneqq X$ and such that Su is isomorphic to $X = Sx$, then we may assume that the isomorphism is given by the map $sx \mapsto su$ for all $s \in S$, picking a different generator for Su if necessary. Let $\{X_n : n = 0, 1, 2, \ldots\}$ be a collection of disjoint copies of X with x_n, $u_n \in X_n$ the elements corresponding to $x, u \in X$. Then the disjoint union $\bigcup \{X_n : n = 0, 1, 2, \ldots\}$ is a left S-operand. We can define a congruence $\sim$ on this operand by setting $sx_n \sim su_{n+1}$ for all $s \in S$ and all $n = 0, 1, \ldots$. Then the operand $Y = (\bigcup \{X_n : n = 0, 1, \ldots\})/\sim$ is the union of a proper ascending chain or orbits, but is not itself an orbit. If we let $y_n \in Y$ be the element of Y corresponding to x_n, then one sees that $Sy_0 \subsetneqq Sy_1 \subsetneqq Sy_2 \subsetneqq \cdots$, that $Y = \bigcup \{Sy_n : n = 0, 1, \ldots\}$, and that Y is not an orbit. Hence S does not satisfy the ascending chain condition on orbits, contrary to hypothesis.

Remark. This construction is most easily illustrated by considering the natural action (translation) of the additive monoid $N = \{0, 1, 2, \ldots\}$ on all the integers.

COROLLARY 3.2.1. *If S is a monoid satisfying the ascending chain condition on orbits, then every cancellative homomorphic image of S is a group.*

Proof. If T is a cancellative homomorphic image of S, then for every $u \in T$ one has T is isomorphic to Tu as a left T-operand, and hence as a left S-operand. Since $Tu \subset T = T1$, it follows from Proposition 3.2 that $Tu = T$. Thus every element of T has a left inverse, which makes T a group.

If S is a monoid, σ a left congruence on S, and $u \in S$, define a left congruence ${}^u\sigma$ by $s\ {}^u\sigma\ t$ if and only if $su\ \sigma\ tu$. Then for $u, v \in S$, ${}^{vu}\sigma = {}^v({}^u\sigma)$. Let $\mathscr{C}_0$ be the set of left congruences on S. If $\sigma \in \mathscr{C}_0$, let $N(\sigma) = \{u \in S : {}^u\sigma = \sigma\}$. Then $N(\sigma)$ is a submonoid of S, σ restricted to $N(\sigma)$ is a congruence, so that one can form the resulting quotient monoid denoted by $N(\sigma)/\sigma$.

Define a relation $\prec$ on $\mathscr{C}_0$ by $\sigma \prec \tau$ for $\sigma, \tau \in \mathscr{C}_0$ if there exists $u \in S$ such that $\sigma = {}^u\tau$. This relation is reflexive and transitive. If we let $\sim$ be the equivalence relation given on $\mathscr{C}_0$ by $\sigma \sim \tau$ if and only if $\sigma \prec \tau$ and $\tau \prec \sigma$, if we set $\mathscr{C} = \mathscr{C}_0/\sim$, and if we denote the relation induced on $\mathscr{C}$ by $\prec$ also by $\prec$, then $(\mathscr{C}, \prec)$ is partially ordered by the reflexive, transitive, antisymmetric relation $\prec$.

THEOREM 3.3. *A monoid S satisfies the ascending chain condition on orbits if and only if*

(i) *$N(\sigma)/\sigma$ is a group for all $\sigma \in \mathscr{C}_0$, and*

(ii) *the partially ordered set $(\mathscr{C}, \prec)$ satisfies the ascending chain condition.*

Proof. We first show that (i) is equivalent to the fact that no left S-orbit contains a proper suborbit isomorphic to itself.

Let $\sigma \in \mathscr{C}_0$ and let $u \in N(\sigma)$. Consider the cyclic left S-operand S/σ and let $[u] \in S/\sigma$ be the class of u. Since $\sigma = {}^u\sigma$, the map $S/\sigma = S[1] \to S[u] \subset S[1]$ given by $[s] \mapsto [su]$ is a well defined isomorphism of S-operands. If no S-orbit has a proper suborbit isomorphic to itself, then $S[u] = S[1]$ so there is a $v \in S$ such that $[1] = v[u] = [vu]$, i.e., such that $vu\ \sigma\ 1$. But now for $s, t \in S$, $s\ {}^v\sigma\ t$ if and only if $sv\ \sigma\ tv$, if and only if $sv\ {}^u\sigma\ tv$ since $\sigma = {}^u\sigma$, if and only if $svu\ \sigma\ tvu$, if and only if $s\ \sigma\ svu\ \sigma\ tvu\ \sigma\ t$ since σ is a left congruence. Hence ${}^v\sigma = \sigma$ so that $v \in N(\sigma)$, and thus every element of the monoid $N(\sigma)/\sigma$ has a left inverse, which makes $N(\sigma)/\sigma$ a group.

On the other hand since every left S-orbit is isomorphic to one of the form S/σ for some $\sigma \in \mathscr{C}_0$ [1], suppose there exists $u \in S$ such that $S[u]$ is isomorphic to $S/\sigma = S[1]$. Again without loss of generality, replacing $[u]$ by another generator of $S[u]$ if necessary, we may assume that this isomorphism $S[1] \to S[u]$ is given by $[s] \mapsto [su]$ for all $s \in S$. Since this is an S-homomorphism, $s\ \sigma\ t$ implies that $su\ \sigma\ tu$, and since it is one-to-one, $su\ \sigma\ tu$ implies that $s\ \sigma\ t$ for $s, t \in S$. This says $\sigma = {}^u\sigma$ so that $u \in N(\sigma)$. If $N(\sigma)/\sigma$ is a group, there exists $v \in N(\sigma)$ such that $vu\ \sigma\ 1$. Hence $[1] = v[u] \in S[u]$, so that $S[u] = S[1] = S/\sigma$, and S/σ has no proper suborbit isomorphic to itself.

We have shown that condition (i) is equivalent to the condition that no S-orbit contains a proper suborbit isomorphic to itself, and therefore, (i) is a necessary condition for S to satisfy the ascending chain condition on orbits.

If (i) and (ii) hold, let X be a left S-operand such that $X = \{Sx_n : n = 0, 1, 2,...\}$ where $Sx_0 \subset Sx_1 \subset Sx_2 \subset \cdots \subset X$. For each n define a left congruence σ_n on S by $s\ \sigma_n\ t$ if and only if $sx_n = tx_n$. Then if $u \in S$ is such that $ux_{n+1} = x_n$, one finds that $\sigma_n = {}^u\sigma_{n+1}$. Hence in $\mathscr{C}_0$, $\sigma_n \prec \sigma_{n+1}$ for all $n = 0, 1,...$. Now $\sigma_n \sim \sigma_{n+1}$ if and only if there also exists $v \in S$ such that $\sigma_{n+1} = {}^v\sigma_n$. But then $\sigma_{n+1} = {}^{vu}\sigma_{n+1}$ so that $vu \in N(\sigma_{n+1})$. By (i) there is a $w \in N(\sigma_{n+1})$ such that $wvu\ \sigma_{n+1}\ 1$. This says $x_{n+1} = wvux_{n+1} = wvx_n$ so that $Sx_n = Sx_{n+1}$. In this way one finds that $\sigma_n \sim \sigma_{n+1}$ in $\mathscr{C}_0$ if and only if $Sx_n = Sx_{n+1}$, so that in $(\mathscr{C}, \prec)$ $\mathrm{cls}(\sigma_n) \neq \mathrm{cls}(\sigma_{n+1})$ if and only if

$Sx_n \neq Sx_{n+1}$. Since by (ii) $(\mathscr{C}, \prec)$ satisfies the ascending chain condition, the chain $\mathrm{cls}(\sigma_0) \prec \cdots \prec \mathrm{cls}(\sigma_n) \prec \mathrm{cls}(\sigma_{n+1}) \prec \cdots$ must terminate. Therefore, so must the chain $Sx_0 \subset \cdots \subset Sx_n \subset Sx_{n+1} \subset \cdots$. Thus S satisfies the ascending chain condition on orbits.

Conversely, if S satisfies the ascending chain condition on orbits, then (i) holds. Suppose $\mathrm{cls}(\sigma_0) \prec \cdots \prec \mathrm{cls}(\sigma_n) \prec \mathrm{cls}(\sigma_{n+1}) \prec \cdots$ is an ascending chain in $(\mathscr{C}, \prec)$. Then let $u_n \in S$ such that ${}^{(u_n)}(\sigma_{n+1}) = \sigma_n$, and denote a typical element of S/σ_n by $[s]_n$ for $s \in S$. One can define a congruence $\approx$ on the disjoint union $\bigcup \{S/\sigma_n : n = 0, 1, 2,...\}$ by $[s]_n \approx [su_n]_{n+1}$ for all $s \in S$ and all $n = 0, 1,...$. Let $Y = (\bigcup \{S/\sigma_n : n = 0, 1,...\})/\approx$. Then letting y_n be the class of $[1]_n$ in Y one finds

$$Sy_0 \subset Sy_1 \subset \cdots \subset Sy_n \subset \cdots \bigcup \{Sy_n : n = 0, 1, 2,...\} = Y, \text{ and } Sy_n \approx S/\sigma_n .$$

Since S satisfies the ascending chain condition on orbits, this chain must terminate. Hence by the remarks in the last paragraph so must the chain in $(\mathscr{C}, \prec)$. Thus if S satisfies the ascending chain condition on orbits, (i) and (ii) must hold.

Remark. If S is a monoid satisfying the descending chain condition on left orbits (i.e., every left S-operand satisfies the descending chain condition on orbits) then "ascending chain condition" may be replaced by "descending chain condition" throughout Theorem 3.3 and the corresponding result remains true. The proof is the same, once one observes that S satisfying the descending chain condition on left orbits also implies that no S-orbit contains a proper suborbit isomorphic to itself.

One easily sees that the class of monoids satisfying the ascending chain condition on orbits includes all groups and all finite monoids. A direct computation shows that it includes $N_{\max} = \{0, 1, 2,...\}$ with $a \cdot b = \max(a, b)$. This example also shows that the ascending chain condition on orbits does not imply the descending chain condition on orbits. However, this class excludes the additive monoid N, and in fact all monoids having such nice cancellation properties as, say, having a cancellative left ideal which is not an $\mathscr{L}$-class or a cancellative homomorphic image which is not a group. If one lets $N_{\min} = \{0, 1, 2,...\} \cup \{I\}$ where I is a two-sided identity and otherwise $a \cdot b = \min(a, b)$, and if one considers $N_{\min}$ acting on $N_{\min}\backslash\{I\}$, one sees that $N_{\min}$ does not satisfy the ascending chain condition on orbits; hence not all torsion monoids satisfy the ascending condition on orbits. A direct computation shows that $N_{\min}$ satisfies the descending chain condition on orbits. Hence the two chain conditions are independent.

If S is a monoid satisfying the ascending chain condition on orbits, then S need not be torsion (since all groups satisfy the ascending chain condition on orbits); however, one can ask whether S must satisfy the following torsion criterion: For every $x \in S$ there exists a subgroup $G \subset S$ and a positive integer k such that $x^k \in G$.

Addendum

Pursuing a suggestion made by Professor K. H. Hofmann, the author has verified the fact that the results of the first two sections of this paper apply as they stand to any partially ordered, or even pre-ordered, set. One need only read "$x \leqslant y$" for "$x \in Sy$" in the above, and one finds that the generalized orbits become dual filters and the maximal generalized orbits become dual ultrafilters. With this translation, the definitions, theorems, and proofs carry over completely to the more general situation.

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Reference

1. A. H. Clifford and G. B. Preston, "The Algebraic Theory of Semigroups," Vol. 2, American Mathematics Society, Providence, RI, 1967.